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$H - C^1$ maps and elliptic SPDEs with non-linear local perturbations of Nelson's Euclidean free field

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Abstract

Elliptic stochastic partial differential equations (SPDE) with polynomial perturbation terms defined in terms of Nelson's Euclidean free field on R^d are studied using results by S. Kusuoka and A.S. Üstünel and M. Zakai concerning transformation of measures on abstract Wiener space. SPDEs of this type arise, in particular, in (Euclidean) quantum field theory with interactions of the polynomial type. The probability laws of the solutions of such SPDEs are given by *Girsanov probability measures*, that are non-linearly transformed measures of the probability law of Nelson's free field defined on subspaces of Schwartz space of tempered distributions.

Introduction

In this paper we study elliptic stochastic partial (pseudo) differential equations (SPDE) heuristically written as follows

$$(-\Delta + 1)\psi(x) + V(\psi)(x) = (-\Delta + 1)^{\frac{1}{2}}\dot{W}(x) \qquad x \in \boldsymbol{R}^d, \tag{1}$$

where Δ is the d-dimensional Laplace operator, V is a (renormarized) polynomial function, and W is an isonormal Gaussian process on $\boldsymbol{R}^d$ (cf.Nualart [10], and for precise definition of $(-\Delta+1)^{\frac{1}{2}}\dot{W}$ see Theorem 1.1). $\dot{W}$ is often referred to as the *Gaussian white noise* on $\boldsymbol{R}^d$.

The existence problem for the solution ψ of (1), as a tempered distribution valued random variable, and the problem of deriving probabilistic properties for the solution, such as characterizing a class of functionals of the solution possessing the so called reflection positivity, will be solved by reducing these problems to the existence problem of the associated Girsanov probability measure and the absolutely continuity of the measure with respect to a reference measure.

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1 Construction of nonlinear $H - C^1$ maps on Nelson's free field

We shall first recall the definition of a stochastic process on a parameter space $\mathcal{D}$ and its equivalent class.

i) Let $\mathcal{D}$ be a locally convex topological vector space (TVS) which is separable, and $(\Omega, \mathcal{F}, P)$ be a complete probability space. A family of complex valued random variables $\{\Psi(\varphi, \omega)\}_{\varphi \in \mathcal{D}}$ on $(\Omega, \mathcal{F}, P)$ is called as a stochastic process with parameter space $\mathcal{D}$.

ii) Two stochastic processes $\{\Psi(\varphi, \omega)\}_{\varphi \in \mathcal{D}}$ and $\{\tilde{\Psi}(\varphi, \omega)\}_{\varphi \in \mathcal{D}}$ on $(\Omega, \mathcal{F}, P)$ are said to be equivalent if

$$\forall \varphi \in \mathcal{D}, \qquad P(\{\omega | \Psi(\varphi, \omega) = \tilde{\Psi}(\varphi, \omega)\}) = 1.$$

iii) Two stochastic processes $\{\Psi(\varphi, \omega)\}_{\varphi \in \mathcal{D}}$ and $\{\tilde{\Psi}(\varphi, \omega)\}_{\varphi \in \mathcal{D}}$ on $(\Omega, \mathcal{F}, P)$ are said to be strongly equivalent if

$$P(\{\omega | \forall \varphi \in \mathcal{D}, \quad \Psi(\varphi, \omega) = \tilde{\Psi}(\varphi, \omega)\}) = 1.$$

Let $\mathcal{S}(\boldsymbol{R}^d)$ be the Schwartz space of rapidly decreasing test functions equipped with usual topology. $\mathcal{S}(\boldsymbol{R}^d)$ is a nuclear space. Let $\mathcal{S}'(\boldsymbol{R}^d)$ be its topological dual.

Let Δ be the d-dimensional Laplacian, and set $J^\alpha = (-\Delta + m^2)^{-\alpha}$ for some fixed $m > 0$. Precisely J^α is the pseudo differential operator with the symbol $(|\xi|^2 + m^2)^{-\alpha}$, $\xi \in \boldsymbol{R}^d$. We denote the kernel representation of J^α by $J^\alpha(x - y)$: $(J^\alpha \varphi)(x) = \int_{\boldsymbol{R}^d} J^\alpha(x - y)\varphi(y)dy$, for $\varphi \in \mathcal{S}$. This is defined by the Fourier inverse transform such that

$$J^\alpha(x) = (2\pi)^{-d} \int_{\boldsymbol{R}^d} e^{\sqrt{-1}x \cdot \xi} (|\xi|^2 + m^2)^{-\alpha} d\xi \in L^1(\boldsymbol{R}^d; \lambda^d).$$

An integral representation of this Green kernel by means of a modified Bessel function, which also puts into evidence its regularity, is well known (cf. for e.g. [15]).

For each a, b, $d > 0$ let $B_d^{a,b}$ be the linear subspace of $\mathcal{S}'(\boldsymbol{R}^d)$ defined by

$$B_d^{a,b} = \{(|x|^2 + 1)^{\frac{b}{4}} J^{-a} f : \ f \in L^2(\boldsymbol{R}^d; \lambda^d)\}, \tag{2}$$

where λ denotes the Lebesgue measure on $\boldsymbol{R}^d$. $B_d^{a,b}$ is a separable Hilbert space with the scalar product

$$< u | v > = \int_{\boldsymbol{R}^d} J^a((|x|^2 + 1)^{-\frac{b}{4}} u(x)) \, J^a((|x|^2 + 1)^{-\frac{b}{4}} v(x)) dx, \qquad u, v \in B_d^{a,b}. \tag{3}$$

Note that if a, b, $d > 0$, then $C_0(\boldsymbol{R}^d) \subset B_d^{a,b}$. From the consideration of cylinder sets constructed from $C_0(\boldsymbol{R}^d)$ and $B_d^{a,b}$ it is easy to see that

$$\mathcal{B}(C_0(\boldsymbol{R}^d \to \boldsymbol{R})) = \left\{ A \cap C_0(\boldsymbol{R}^d \to \boldsymbol{R}) \ : \ A \in \mathcal{B}(B_d^{a,b}) \right\}, \tag{4}$$

where $\mathcal{B}(C_0(\boldsymbol{R}^d \to \boldsymbol{R}))$ and $\mathcal{B}(B_d^{a,b})$ are the Borel σ-fields of $C_0(\boldsymbol{R}^d)$ and $B_d^{a,b}$ respectively (this is obvious because the Borel σ field of a locally convex topological vector space which is separable is generated by its cylinder sets, cf. Yoshida [15]).

We use the same terminology and notations concerning multiple stochastic integrals, abstract Wiener spaces and transformations between abstract Wiener spaces which are used in [10] and [14].

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and consider an isonormal Gaussian process $W = \{W(h), h \in L^2_{real}(\boldsymbol{R}^d; \lambda^d)\}$, where λ^d denotes the Lebesgue measure on $\boldsymbol{R}^d$ and L^2_{real} is the real L^2 space: W is a centered Gaussian family of random variables on $(\Omega, \mathcal{F}, P)$ such that

$$E[W(h)W(g)] = \int_{\boldsymbol{R}^d} h(x)\, g(x) \lambda^d(dx), \qquad h,\ g \in L^2_{real}(\boldsymbol{R}^d; \lambda^d),$$

where E denotes the expectation with respect to the probability measure P. Ω can be taken to be the complete separable metric space $\boldsymbol{R}^\infty$ equipped with the metric

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} 2^{-n} \min\{|x_n - y_n|, 1\},\ \boldsymbol{x} = (x_1, x_2, x_3, \cdots), \boldsymbol{y} = (y_1, y_2, y_3, \cdots),$$

$$P = N_{0,1}^{\infty} \tag{5}$$

and $\mathcal{F}$ to be the completion of the Borel σ-field of Ω with respect to P.

For $A \in \mathcal{B}(\boldsymbol{R}^d)$ such that $\lambda^d(A) < \infty$ we set

$$W(A) = W(\chi_A), \qquad \text{where } \chi_A \text{ is the indicator function of the set } A.$$

Then, for $h \in L^2_{real}(\boldsymbol{R}^d; \lambda^d)$ the random variable $W(h)$ can be regarded as a stochastic integral, and is denoted by

$$W(h) = \int_{\boldsymbol{R}^d} h\, dW.$$

In the sequel we sometimes use the notation $W(\varphi) =< \varphi, \dot{W} >_{\mathcal{S},\mathcal{S}'}$ for $\varphi \in \mathcal{S}$. The multiple stochastic integrals, such as (12) below, are defined in the usual way. Namely a multiple stochastic integral is the limit of a sequence of multiple sums of Gaussian random variables such that $\sum_{i_1,\ldots,i_p} a_{i_1,\ldots,i_p} W(A_{i_1}) \times \cdots \times W(A_{i_p})$, where $a_{i_1,\ldots,i_p} = 0$ if $i_j = i_k$ for some $j \neq k$ (i.e. by taking sums with eliminaton of all diagonal parts), for a precise definition of multipe stochastic integral cf. section 1.1.2 of [10].

We denote the Fourier and Fourier inverse transform of a function φ respectively by $\mathcal{F}[\varphi]$ and $\mathcal{F}^{-1}[\varphi]$, which are defined by

$$\mathcal{F}[\varphi](\xi) = \int_{\boldsymbol{R}^d} e^{-\sqrt{-1}x\cdot\xi} \varphi(x) dx,$$

$$\mathcal{F}^{-1}[\varphi](\xi) = (2\pi)^{-d} \int_{\boldsymbol{R}^d} e^{\sqrt{-1}x\cdot\xi} \varphi(x) dx \qquad \text{for} \quad \varphi \in \mathcal{S}(\boldsymbol{R}^d).$$

We sometimes denote $\mathcal{F}[\varphi] = \hat{\varphi}$. Let $\eta_1 \in C_0^\infty(\boldsymbol{R}^d)$ be such that $\eta_1(x) = \eta_1(y)$ for $|x| = |y|$ and

$$0 \le \eta_1(x) \le 1, \qquad \eta_1(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| \ge 2, \end{cases} \tag{6}$$

and let $\eta_k(x) = \eta_1(\frac{x}{k}) \in C_0^\infty(\boldsymbol{R}^d)$, $k = 1, 2, 3, \ldots$. Also define $\rho \in C_0^\infty(\boldsymbol{R}^d)$ as follows:

$$\rho(x) = \begin{cases} C \exp(-\dfrac{1}{1 - |x|^2}) & |x| < 1 \\ 0 & |x| \ge 1 \end{cases},$$

where the constant C is taken to satisfy

$$\int_{\boldsymbol{R}^d} \rho(x)dx = 1. \tag{7}$$

Let

$$\rho_k(x) = k^d \rho(kx), \qquad k = 1, 2, 3, \ldots.$$

For $\alpha > 0$ we define $J_k^\alpha \in \mathcal{S}(\boldsymbol{R}^d)$, $k = 1, 2, 3, \ldots$ by

$$J_k^\alpha(x) = \int_{\boldsymbol{R}^d} J^\alpha(y)\rho_k(x-y)dy. \tag{8}$$

Also

$$F_k^\alpha(x; y_1, \ldots, y_p) = (\eta_k(x))^p J_k^\alpha(x-y_1)\cdots J_k^\alpha(x-y_p), \tag{9}$$

and

$$F^\alpha(x; y_1, \ldots, y_p) = J^\alpha(x-y_1)\cdots J^\alpha(x-y_p), \quad p = 1, 2, 3, \ldots. \tag{10}$$

Then we see that the function F_k^α and F^α are symmetric in the last p variables $(y_1, \ldots, y_p)$ and

$$F_k^\alpha \in \mathcal{S}((\boldsymbol{R}^d)^{p+1}), \qquad F_k^\alpha(x; y_1, \ldots, y_p) = 0 \quad \text{for} \quad |x| \geq 2k. \tag{11}$$

For each $\alpha > 0$, $p \geq 1$ and $k \geq 1$ we define the random variable $:_k \phi_{\alpha,\omega}^p :$ as a multiple stochastic integral such that

$$:_k \phi_{\alpha,\omega}^p : (x) = \int_{(\boldsymbol{R}^d)^p} F_k^\alpha(x; y_1, \ldots, y_p) dW_\omega(y_1)\cdots dW_\omega(y_p). \tag{12}$$

Remark 1. 1 (continuous version of $:_k \phi_\alpha^p :$) *For each fixed $k \in \boldsymbol{N}$ it is easy to see that $\{:_k \phi_{\alpha,\omega}^p : (x)\}_{x\in\boldsymbol{R}^d}$ satisfies the Kolmogorov's continuity criterion for processes on $\boldsymbol{R}^d$ (cf., e.g., Section A.3 of [10]), and has an equivalent process $\{:_k \tilde{\phi}_{\alpha,\omega}^p : (x)\}_{x\in\boldsymbol{R}^d}$ which is a $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variable:*

$$P(:_k \phi_{\alpha,\omega}^p : (x) =:_k \tilde{\phi}_{\alpha,\omega}^p : (x)) = 1, \qquad \forall x \in \boldsymbol{R}^d,$$

$$P(:_k \tilde{\phi}_{\alpha,\omega}^p :\in C_0(\boldsymbol{R}^d \to \boldsymbol{R})) = 1.$$

We always take $\{:_k \phi_{\alpha,\omega}^p : (x)\}_{x\in\boldsymbol{R}^d}$ as its continuous modification $\{:_k \tilde{\phi}_{\alpha,\omega}^p : (x)\}_{x\in\boldsymbol{R}^d}$ and drop the tilde in the following. Then by (4) $\{:_k \phi_{\alpha,\omega}^p : (x)\}_{x\in\boldsymbol{R}^d}$ is understood as a $B_d^{a,b}$ (a, $b \geq 0$) valued random variable on $(\Omega, \mathcal{F}, P)$.

Theorem 1. 1

Suppose that the positive integer p and the positive real numbers a, b and α satisfy

$$\min\left(1, \frac{4a}{d}\right) + p \times \min\left(1, \frac{4\alpha}{d}\right) > p, \qquad b > d. \tag{13}$$

Then $\{:_k \phi_{\alpha,\omega}^p :\}_{k\in\boldsymbol{N}}$ is a Cauchy sequence in $L^2(\Omega \to B_d^{a,b}; P)$ (cf. Remark 1.1) and there exists a $B_d^{a,b}$-valued random variable $: \phi_{\alpha,\omega}^p :\in L^2(\Omega \to B_d^{a,b}, P)$ such that

$$\lim_{k\to\infty} \int_\Omega \left\| :_k \phi_{\alpha,\omega}^p : - : \phi_{\alpha,\omega}^p : \right\|_{B_d^{a,b}}^2 P(d\omega) = 0, \tag{14}$$

$$P\left(<: \phi^{p}_{\alpha,\omega} :, \varphi>_{\mathcal{S}',\mathcal{S}} = l_{p,\omega}(\varphi)\right) = 1, \qquad \forall\varphi \in \mathcal{S}(\boldsymbol{R}^d), \tag{15}$$

where

$$l_{p,\omega}(\varphi) = \int_{(\boldsymbol{R}^d)^p} \Big(\int_{\boldsymbol{R}^d} \varphi(x)\, J^{\alpha}(x-y_1) \cdots J^{\alpha}(x-y_p)\, dx \Big) dW_{\omega}(y_1) \cdots dW_{\omega}(y_p). \tag{16}$$

The proof of Theorem 1.1 has been given by Yoshida [15]. By Remark 1.1 and (4), since the $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variable $:_k \phi^{p}_{\alpha,\omega}:$ can be understood as a $B^{a,b}_d$ (a, $b > 0$)-valued random variable by making use of its multiple stochastic integral expression, it is easy to see that this random variable is in $L^2(\Omega \to B^{a,b}_d; P)$. Then by making use of a Fubini type theorem concerning the stochastic integral resp. Lebesgue integral on $\boldsymbol{R}^d$, the theorem follows. ■

In the sequel we shall denote $:_k \phi^{1}_{\alpha,\omega}:$ and $:\phi^{1}_{\alpha,\omega}:$ by ${}_k\phi_{\alpha,\omega}$ and $\phi_{\alpha,\omega}$ respectively. In particular when $\alpha = \frac{1}{2}$, then for each given d the $\mathcal{S}'(\boldsymbol{R}^d)$-valued random variable (cf. Theorem 1.1) $\phi_{\frac{1}{2},\omega}$ is a stochastic integral expression for Nelson's free *Euclidean* field, we denote it simply by ϕ_{ω} and we write

$$\phi_{\omega} = J^{\frac{1}{2}} \dot{W}_{\omega}.$$

Now, by making use of the above results and notations let us study non-linear shifts on Nelson's free field in the context of abstract Wiener spaces. For given d, let μ be the probability law of $\phi_{\omega} = \phi_{\frac{1}{2},\omega}$. Since ϕ_{ω} is a $B^{a,b}_d$-valued random variable ($a > \frac{d-2}{4}$, $b > d$ by Theorem 1.1) on $(\Omega, \mathcal{F}, P)$, μ is a probability measure on $B^{a,b}_d$:

$$\mu(A) = P(\{\omega | \phi_{\omega} \in A\}), \quad A \in \mathcal{B}(B^{a,b}_d) \quad (a > \frac{d-2}{4},\ b > d). \tag{17}$$

We remark that for the complete probability space $(\Omega, \mathcal{F}, P)$ defined by (5), the following holds: If we let

$$\mathcal{B}^{\mu} = \{A | \{\omega | \phi_{\omega} \in A\} \in \mathcal{F}\},$$

then the probability space $(B^{a,b}_d, \mathcal{B}^{\mu}, \mu)$ is a complete probability space, i.e.

$$\mathcal{B}^{\mu} = \overline{\mathcal{B}(B^{a,b}_d)}^{\mu} = \text{the completion of } \mathcal{B}(B^{a,b}_d) \text{ with respect to } \mu. \tag{18}$$

Hence, the map τ_k defined by (21) below is a $B^{a',b}_d$-valued random variable on $(B^{a,b}_d, \mathcal{B}^{\mu}, \mu)$.

Theorem 1. 2 *Suppose that a, β, a', p and b satisfy*

$$\min(1, \frac{4a}{d}) + \min(1, \frac{2}{d}) > 1, \tag{19}$$

$$\min(1, \frac{4a'}{d}) + p \times \min(1, \frac{4\beta}{d}) > p, \qquad b > d. \tag{20}$$

For each k let $\tau_k = \tau_{(\beta,p),k}$ be the measurable map from $B^{a,b}_d$ to $B^{a',b}_d$ defined by

$$\tau_k(\psi)(x) = p!(\eta_k(x))^p \sum_{n=0}^{[\frac{p}{2}]} \frac{(-\frac{1}{2}c_{\beta,k})^n}{n!(p-2n)!} \Big(< J^{\beta}_k(x-\cdot), (J^{-\frac{1}{2}}\psi)(\cdot) >_{\mathcal{S},\mathcal{S}'} \Big)^{p-2n}, \quad \textit{for } \psi \in B^{a,b}_d, \tag{21}$$

where

$$c_{\beta,k} = \int_{\boldsymbol{R}^d} (J^{\beta}_k(y))^2 dy.$$

Then

$$P\left(\{\omega \mid \tau_k(\phi_\omega)(x) =:_k \phi^p_{\beta,\omega} : (x) \quad \forall x \in \boldsymbol{R}^d\}\right) = 1, \tag{22}$$

the $B_d^{a',b}$-valued measurable functions $\{\tau_k(\psi)\}$ on $(B_d^{a,b}, \mathcal{B}^\mu, \mu)$ form a Cauchy sequence in the Banach space $L^2(B_d^{a,b} \to B_d^{a',b}; \mu)$, and there exists a $\mathcal{B}(B_d^{a',b})/\mathcal{B}^\mu$-measurable function $\tau = \tau_{(\beta,p)} \in L^2(B_d^{a,b} \to B_d^{a',b}; \mu)$ such that

$$\lim_{k\to\infty} \int_{B_d^{a,b}} \|\tau_k(\psi) - \tau(\psi)\|^2_{B_d^{a',b}} \mu(d\psi) = 0, \tag{23}$$

or equivalently

$$\lim_{k\to\infty} \int_{\Omega} \|\tau_k(\phi_\omega) - \tau(\phi_\omega)\|^2_{B_d^{a',b}} P(d\omega) = 0. \tag{24}$$

Moreover one has

$$\tau(\phi_\omega) =: \phi^p_{\beta,\omega} : \qquad P - a.s. \quad \omega \in \Omega. \tag{25}$$

By the definition of Wick power and multiple stochastic integral (22) can easily be proved. The existence of τ is proved by using Theorem 1.1 and (22), these proofs have been given in [15] ■

Next, we shall see that Nelson's Euclidean free field possesses the structure of an abstract Wiener space, and then show that the map $\tau_{(\beta,p)}$ on the abstract Wiener space has sufficient regularity.

As usual let $H^\gamma = H^\gamma(\boldsymbol{R}^d)$ be the Sobolev space on $\boldsymbol{R}^d$ such that

$$H^\gamma(\boldsymbol{R}^d) = \left\{\phi \in \mathcal{S}'(\boldsymbol{R}^d) \mid \int_{\boldsymbol{R}^d} |\mathcal{F}\phi|^2(x)(1+|x|^2)^\gamma dx < \infty\right\}.$$

In order to make the notations simple, we equip $H^\gamma(\boldsymbol{R}^d)$ with the inner product

$$< u, v >_{H^\gamma} = (2\pi)^{-d} \int_{\boldsymbol{R}^d} (\mathcal{F}u)(x)\,(\mathcal{F}v)(x)\,(m^2+|x|^2)^\gamma dx$$

for a given constant $m > 0$ (interpreted as "mass parameter").
Then by Theorem 1.1 for $a > \frac{d}{4} - \frac{1}{2}$ we see that $(B_d^{a,b}, \mu)$ is an abstract Wiener space and one has, for $\varphi \in \mathcal{S}(\boldsymbol{R}^d)$:

$$\begin{aligned}
&\int_{B_d^{a,b}} e^{\sqrt{-1}<\psi,\varphi>_{\mathcal{S}',\mathcal{S}}} \mu(d\psi) \\
&\quad = \int_\Omega \exp\left[\sqrt{-1}\int_{\boldsymbol{R}^d}\left(\int_{\boldsymbol{R}^d} \varphi(x) J^{\frac{1}{2}}(x-y)dx\right)dW_\omega(y)\right] P(d\omega) \\
&\quad = \exp\left(-\frac{1}{2}\|\varphi\|^2_{H^{-1}}\right) = \exp\left(-\frac{1}{2}\|J^1\varphi\|^2_{H^1}\right).
\end{aligned} \tag{26}$$

The inclusion map $i : H^{-1} \to B_d^{a,b}$ defined by

$$i(h) = J^1 h, \qquad h \in H^{-1} \tag{27}$$

is continuous and $i(H^{-1}) = H^1$ is dense in $B_d^{a,b}$. By this we can identify H^{-1} with H^1, and we have the following continuous injection:

$$(B_d^{a,b})^* \hookrightarrow H^{-1} \cong H^1 \hookrightarrow B_d^{a,b}.$$

$$\mathcal{H} = H^{-1}$$

we will consider the abstract Wiener space $(B_d^{a,b}, i(\mathcal{H}), \mu)$ with Cameron-Martin space

$$i(\mathcal{H}) = J^1 H^{-1} = H^1. \tag{28}$$

We then apply the results given by [7], [14] concerning the (non-linear) shifts on Wiener spaces to the maps τ defined above.

Remark 1. 2 *Nelson's Euclidean free field is defined originally as a Gaussian process indexed by $\mathcal{H} = H^{-1}$ (cf. Nelson [8]), i.e. Gaussian process with the index set H^{-1} of which characteristic function is*

$$\exp(-\frac{1}{2}||\varphi||^2_{H^{-1}}), \qquad \varphi \in H^{-1} \quad (cf.\ (26)).$$

By this, here we prefer to denote the Cameron-Martin space by $i(\mathcal{H})$, and denote the abstract Wiener space by $(B_d^{a,b}, i(\mathcal{H}), \mu)$. Then our calculus on the abstract Wiener space will be performed through $\mathcal{H}$.

Definition 1. 1 (Representative for $\tau_{(\beta,p)}$) *For each p by (23) we can take subsequences $\{\tau_{(\beta,1),k_j}\}, \ldots, \{\tau_{(\beta,p),k_j}\}$ and a set $B(\beta,p) \in \mathcal{B}^\mu$ satisfying $\mu(B(\beta,p)) = 1$ such that*

$$\lim_{k_j \to \infty} ||\tau_{(\beta,q),k_j}(\psi) - \tau_{(\beta,q)}(\psi)||^2_{B_d^{a,b}} = 0, \qquad \forall \psi \in B(\beta,p) \quad q = 1, \ldots, p.$$

We denote by $\overline{B}(\beta,p)$ the set of all $\psi \in B_d^{a,b}$ such that the limits $\lim_{k_j \to \infty} \tau_{(\beta,q),k_j}(\psi)$ exist, $q = 1, \ldots, p$, in $B_d^{a',b}$ for some $a \le a'$. Then $\overline{B}(\beta,p)$ is $\mathcal{B}^\mu$-measurable. In the sequel we fix a representative $\overline{\tau}_{(\beta,p)}$ of $\tau_{(\beta,p)}$ defined as follows:

$$\overline{\tau}_{(\beta,p)} = \begin{cases} \lim_{k_j \to \infty} \tau_{(\beta,p),k_j}(\psi) & \psi \in \overline{B}(\beta,p) \\ 0 & \textit{elsewhere.} \end{cases}$$

$\overline{\tau}_{(\beta,p)}$ will be simply denoted by $\tau_{(\beta,p)}$.

Theorem 1. 3 (polynomial $H - C^1$ map) *Let $b > d$ and a be a number such that $a > \frac{d}{4} - \frac{1}{2}$. Let $(B_d^{a,b}, i(\mathcal{H}), \mu)$ be the abstract Wiener space defined above, and denote the "Gross-Sobolev derivative" and "divergence" operators on $(B_d^{a,b}, i(\mathcal{H}), \mu)$ by ∇ and δ, respectively (cf. [7], [10], [14]). For $M \ge 0$ let η_M be the space-cut-off such that $\eta_M(x) = \eta_1(\frac{x}{M})$ (cf. (6)).*

1°) *Let the integer p and the real number $\beta > 0$ satisfy*

$$\beta > \frac{d}{4}\frac{p+1}{p+2}. \tag{29}$$

Then the map $u_p(\psi) = J^{\beta - \frac{1}{2}}(\eta_M \tau_{(\beta,p)}(\psi))$ ($\mathcal{H}$-valued Wiener functional) is an element of $\boldsymbol{D}_{2,k}(\mathcal{H})$ ($\forall k \ge 1$), and the following holds:

$$\begin{aligned} \nabla u_p(\psi)(x,y) &= p\Big\langle \eta_M, \tau_{(\beta,p-1)}(\psi)(\cdot) J^{\beta-\frac{1}{2}}(\cdot - x) J^{\beta - \frac{1}{2}}(\cdot - y)\Big\rangle_{S,S'} \\ &\in L^2(\mathcal{H} \otimes \mathcal{H}; \mu). \end{aligned}$$

Let $B(\beta,p)$ be as in Definition 1-ii) for these p and β, then $\mu(B(\beta,p)) = 1$ and $B(\beta,p) + H^1 \subset \overline{B}(\beta,p)$.
The divergence of u_p is given by

$$\delta u_p(\psi) = <\eta_M, \tau_{(\beta,p+1)}(\psi)>_{S,S'} \qquad \mu - a.s. \quad \psi \in B_d^{a,b}. \tag{30}$$

2°) *If*

$$\beta > \frac{d}{4}\left(\frac{p-2}{p-1} + \frac{2}{3(p-1)}\right) \tag{31}$$

(which is a particular case of $i - 1°$)), then

$$\begin{aligned}&\nabla u_p(\psi + i(h))(x,y)\\ &= p\sum_{q=0}^{p-1}\binom{p-1}{q}\Big\langle \eta_M, (J^{\beta-\frac{1}{2}}(i(h)))^q \tau_{(\beta,p-1-q)}(\psi)(\cdot)\\ &\times J^{\beta-\frac{1}{2}}(\cdot - x)J^{\beta-\frac{1}{2}}(\cdot - y)\Big\rangle_{S,S'}, \qquad \forall\psi \in B(\beta,p), \quad \forall h \in \mathcal{H}.\end{aligned} \tag{32}$$

u_p is an $H - C^1$ map on $(B_d^{a,b}, i(\mathcal{H}), \mu)$ (cf. [7], [10], [14]):

$$\mathcal{H} \ni h \longmapsto \nabla u_p(\psi + i(h)) \in \mathcal{H} \otimes \mathcal{H} \quad \textit{is continuous for all } \psi \in B(\beta,p). \tag{33}$$

Definition 1. 2 *For $u \in \boldsymbol{D}_{2,1}(\mathcal{H})$ and $\lambda \in \boldsymbol{R}$ we define*

$$\Lambda_{\lambda u}(\psi) = \det{}_2(I_{\mathcal{H}} + \lambda\nabla u(\psi))\exp(-\lambda\delta u(\psi) - \frac{\lambda^2}{2}|u(\psi)|_{\mathcal{H}}^2), \tag{34}$$

where $\det_2(I_{\mathcal{H}} + \lambda\nabla u(\psi))$ denotes the Carleman-Fredholm determinant of the Hilbert-Schmidt operator $\lambda\nabla u(\psi) \in \mathcal{H} \otimes \mathcal{H}$ and $|\quad|_{\mathcal{H}}$ denotes the norm of the Hilbert space $\mathcal{H}$.

2 Main results for SPDE with cubic perturbation

In this section we shall consider elliptic SPDE on $\boldsymbol{R}^d$ formally given by

$$(-\Delta + m^2)\psi(x) + \lambda\eta_M(x) : \psi^3(x) := (-\Delta + m^2)^{\frac{1}{2}}\dot{W}(x), \ x \in \boldsymbol{R}^d. \tag{35}$$

where $\eta_M(x) = \eta_1(\frac{x}{M})$ is the "space-cut-off" defined by (6), and W is an isonormal Gaussian process on $\boldsymbol{R}^d$. Using the measurable maps defined by Theorem 1.2 and Definition 1.1, the above SPDE can be written in the following form:

$$(-\Delta + m^2)\psi(x) + \lambda\eta_M(x)\tau_{(\frac{1}{2},3)}(\psi)(x) = (-\Delta + m^2)^{\frac{1}{2}}\dot{W}(x), \qquad x \in \boldsymbol{R}^d. \tag{36}$$

We reduce the existence problem of the solution of (36) to the existence of corresponding *Girsanov measures*. We shall adopt the notion of "Girsanov measure" given in section 1.3 of [14] for our problem as follows. Let S be a topological space and $\mathcal{B}(S)$ be its Borel σ-field. Let μ be a complete probability measure on $(S, \overline{\mathcal{B}(S)}^\mu)$, and let T be a measurable map such that $T : (S, \overline{\mathcal{B}(S)}^\mu) \longmapsto (S, \mathcal{B}(S))$, where $\overline{\mathcal{B}(S)}^\mu$ = "the completion of $\mathcal{B}(S)$ with respect to μ". A signed measure ν on
$(S, \overline{\mathcal{B}(S)}^\mu)$ will be called as a "*Girsanov measure on $(S, \overline{\mathcal{B}(S)}^\mu)$ associated with μ and T*" if and only if it satisfies

$$\int_S f(T\phi)d\nu(\phi) = \int_S f(\phi)d\mu(\phi)$$

for any bounded measurable $f : (S, \mathcal{B}(S)) \longmapsto (\boldsymbol{R}, \mathcal{B}(\boldsymbol{R}))$. (37)

In particular if such a signed measure ν is a probability measure on $(S, \overline{\mathcal{B}(S)}^{\mu})$, then this will be called the "*Girsanov probability measure on* $(S, \overline{\mathcal{B}(S)}^{\mu})$ *associated with* μ *and* T".

Remark 2. 1 *i) If a "Girsanov probability measure* ν *on* $(S, \overline{\mathcal{B}(S)}^{\mu})$ *associated with* μ *and* T *" exists, then by (37) the probability law of* $T\phi$ *under* ν *is* μ*. In other words, for a random variable* ϕ *taking values in* S *with probability law* ν *there exists a random variable* ψ *with probability law* μ*, and the following holds:*

$$T\phi = \psi.$$

In case ν *is not a probability measure but a signed Girsanov measure on* $(S, \overline{\mathcal{B}(S)}^{\mu})$ *associated with* μ *and* T*, if we set* $\mathcal{B}_T \equiv \{T^{-1}A | A \in \mathcal{B}(S)\}$*, and restrict* ν *to* $\mathcal{B}_T$*, then* $\nu|_{\mathcal{B}_T}$ *is a probability measure on* $(S, \mathcal{B}_T)$ *and the probability law of* $T\phi$ *under* ν *is* μ*. Such signed measures may be important to be considered in relation with the indefinite metric quantum field theory (cf. Albeverio, Gottschalk and Wu [1]).*

Let μ be the probability law of Nelson's free field ϕ on $\boldsymbol{R}^d$, then μ is a complete probability measure on $(B_d^{a,b}, \mathcal{B}^\mu)$ (cf. (18)). Let T be the map defined on $B_d^{a,b}$ such that

$$T(\psi) = \psi + J^1(\lambda \eta_M \tau_{(\frac{1}{2},3)}(\psi)) \qquad \psi \in B_d^{a,b}.$$

We may set $S = B_d^{a,b}$ and $\mathcal{B}(S) = \mathcal{B}(B_d^{a,b})$ in the above general discussion. If there exists ν which is a "*Girsanov probability measure on* $(B_d^{a,b}, \mathcal{B}^\mu)$ *associated with* μ *and* T", then for a $B_d^{a,b}$-valued random variable ψ with probability law ν there exists a Nelson's free field ϕ on $\boldsymbol{R}^d$ and the following holds:

$$\psi + J^1(\lambda \eta_M \tau_{(\frac{1}{2},3)}(\psi)) = \phi.$$

Since ϕ can be expressed by $\phi = J^{\frac{1}{2}} \dot{W}$ for some isonormal Gaussian process W on $\boldsymbol{R}^d$, in the sense of distribution valued random variables this equation means that

$$(-\Delta + m^2)\psi(x) + \lambda \eta_M(x) \tau_{(\frac{1}{2},3)}(\psi)(x) = (-\Delta + m^2)^{\frac{1}{2}} \dot{W}(x), \qquad x \in \boldsymbol{R}^d. \tag{38}$$

By this way we can reduce the existence problem of the solution of the SPDE (38) to the existence problem of the corresponding *Girsanov probability measure.*

In general we give the following definition

Definition 2. 1 (Solution of SPDE) *For given* d *let* $(B_d^{a,b}, i(\mathcal{H}), \mu)$ *be the abstract Wiener space, which is Nelson's Euclidean free field, defined in section 1. For an* $\mathcal{H}$ *valued* $\mathcal{B}^\mu$*-measurable function* $u : B_d^{a,b} \longmapsto \mathcal{H}$ *and for some* $\lambda \in \boldsymbol{R}$ *(note that by Theorem 1.3* $u(\psi) = \eta_M \tau_{(\beta,p)}(\psi)$ *and* $u(\psi) = \eta_M \tau_{(\beta,e^\epsilon)}(\psi)$ *satisfy this measurability condition) set*

$$T(\psi) = \psi + \lambda J^1(u(\psi)), \qquad \psi \in B_d^{a,b}.$$

We say that a probability measure ν *on* $(B_d^{a,b}, \mathcal{B}^\mu)$ *gives a solution of the SPDE*

$$(-\Delta + m^2)\psi(x) + \lambda u(\psi)(x) = (-\Delta + m^2)^{\frac{1}{2}} \dot{W}(x) \qquad x \in \boldsymbol{R}^d,$$

where W *is an isonormal Gaussian process on* $\boldsymbol{R}^d$*, if and only if* ν *is a Girsanov probability measure on* $(B_d^{a,b}, \mathcal{B}^\mu)$ *associated with* μ *and* T.

Lemma 2. 1 (Key lemma for the cubic power perturbation) *Let $d \geq 2$ be given, and suppose that the assumptions of Theorem 1.3-1°) hold for $p = 3$. Also take the numbers $\lambda > 0$ and $\epsilon > 0$ to satisfy $\lambda(1+\epsilon) < \frac{2}{9L}$, where $L = \int_{\mathbf{R}^d} (J^{2\beta}(x))^2 dx$. Then for*

$$u(\psi) = u_3(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,3)}(\psi))$$

defined by Theorem 1.3-1°), the following holds

$$\exp\{-\lambda\delta u + \frac{1+\epsilon}{2}\lambda^2 \|\nabla u\|_2^2\} \in \cap_{q<\infty} L^q(\mu), \tag{39}$$

where $\| \ \|_2$ denotes the Hilbert-Schmidt norm $\| \ \|_{\mathcal{H}\otimes\mathcal{H}}$.

By making use of the fact that δu and ∇u are the 4-th and 2nd Wick power of ψ respectively, this lemma can be proved by applying Nelson's exponential bounds:

Proof of Lemma 2.1 We will prove (39):

$$\exp\{-\lambda\delta u_3 + \frac{1+\epsilon}{2}\lambda^2 \|\nabla u_3\|_2^2\} \in \cap_{q<\infty} L^q(\mu). \tag{40}$$

For simplicity we will give a proof only for the case $d = 2$ and $\beta = \frac{1}{2}$.

The proof will be perfomed by following a strategy given by Nelson (cf. Simon [13]). Namely, let

$$V \equiv -\lambda\delta u_3 + \frac{1+\epsilon}{2}\lambda^2 \|\nabla u_3\|_2^2 \qquad \text{and} \qquad V_k \equiv -\lambda\delta u_{3,k} + \frac{1+\epsilon}{2}\lambda^2 \|\nabla u_{3,k}\|_2^2,$$

where

$$u_{3,k}(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,3),k}(\psi)).$$

Suppose that we can show that there exist κ_1, κ_2 and α that do not depend on k such that

$$V_k(\psi) \leq \kappa_1 (c_k)^2 \qquad \forall k, \quad \mu - a.s.\ \psi \in B_2^{a,b}, \tag{41}$$

$$(E^\mu[|V_k - V|^q])^{\frac{1}{q}} \leq \kappa_2 (q-1)^2 k^{-\alpha}, \qquad q \geq 2, \tag{42}$$

where $c_k = c_{\frac{1}{2},k} = \int_{\mathbf{R}^2} (J_k^{\frac{1}{2}}(y))^2 dy$, defined in Theorem 1.2. Then through the same discussion as Lemma V.5 of [13], we see that there exist $\alpha' > 0$ and $\beta > 0$, independent of k, such that

$$\mu\{\psi \,|\, V \geq \beta(\log k)^2\} \leq e^{-k^{\alpha'}}, \qquad \text{for all large } k.$$

(40) easily follows from this inequality (cf. Theorem V.7 of [13]).

Hence, it suffices to show that (41) and (42) hold for our exponent.

(41) can be shown as follows. For $\psi \in B_2^{a,b}$ let $\psi_k(z) \equiv < J_k^{\frac{1}{2}}(z - \cdot), (J^{-\frac{1}{2}}\psi)(\cdot) >_{S,S'}$, then by (21)

$$\tau_{(\frac{1}{2},2),k}(\psi)(z) = 2!(\eta_k(z))^2 \{\frac{1}{2!}(\psi_k(z))^2 - \frac{1}{2}c_k\},$$

by this we see that

$$\frac{1+\epsilon}{2}(3\lambda)^2 \| < \eta_M(\cdot), \tau_{(\frac{1}{2},2),k}(\psi)(\cdot) J^0(\cdot - x) J^0(\cdot - y) >_{S,S'} \|_{\mathcal{H}\otimes\mathcal{H}}^2$$
$$= \frac{(1+\epsilon)(3\lambda)^2}{2} \int_{\mathbf{R}^2 \times \mathbf{R}^2} \eta_M(z)(\eta_k(z))^2 \{(\psi_k(z))^2 - c_k\} \eta_M(z')(\eta_k(z'))^2$$

$$\times \{(\psi_k(z'))^2 - c_k\}(J^1(z-z'))^2 dzdz'$$
$$= -\frac{(1+\epsilon)(3\lambda)^2}{4} \int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} \Big[\eta_M(z)(\eta_k(z))^2\{(\psi_k(z))^2 - c_k\}$$
$$- \eta_M(z')(\eta_k(z'))^2\{(\psi_k(z'))^2 - c_k\}\Big]^2 (J^1(z-z'))^2 dzdz'$$
$$+ \frac{(1+\epsilon)(3\lambda)^2 L}{2} \int_{\boldsymbol{R}^2} (\eta_M(z))^2(\eta_k(z))^4\{(\psi_k(z))^2 - c_k\}^2 dz, \tag{43}$$
$$\forall \psi \in B_2^{a,b}, \quad \text{where} \quad L = \int_{\boldsymbol{R}^2} (J^1(z))^2 dz.$$

On the other hand, from (21)

$$-\lambda\delta u_3(\psi) = -\lambda < \eta_M, \tau_{(\frac{1}{2},4),k}(\psi) >$$
$$= -\lambda \int_{\boldsymbol{R}^2} \eta_M(z)\Big[4!(\eta_k(z))^4\{\frac{1}{4!}(\psi_k(z))^4 - \frac{\frac{1}{2}c_k}{2!}(\psi_k(z))^2 + \frac{(\frac{1}{2}c_k)^2}{2!}\}\Big]dz,$$
$$\forall \psi \in B_2^{a,b}. \tag{44}$$

Since the first term of the RHS of (43) can not be positive, from (43) and (44) we have the evaluation

$$-\lambda\delta u_k(\psi) + \frac{1+\epsilon}{2}\lambda^2 \|\nabla u_k(\psi)\|^2_{\mathcal{H}\otimes\mathcal{H}}$$
$$\leq \lambda \int_{\boldsymbol{R}^2} \eta_M(z)(\eta_k(z))^4\Big\{-(\psi_k(z))^4 + 6c_k(\psi_k(z))^2 - 3(c_k)^2$$
$$+\frac{3^2(1+\epsilon)}{2}\lambda L\eta_M(z)((\psi_k(z))^4 - 2c_k(\psi_k(z))^2 + (c_k)^2)\Big\}dz,$$
$$\forall \psi \in B_2^{a,b}. \tag{45}$$

Since $0 \leq \eta_M(z) \leq 1$, if ϵ and λ satisfy $\frac{3^2(1+\epsilon)}{2}\lambda L < 1$, then the term in the bracket of the RHS of (45), the biquadratic formula of ψ_k, can not be greater than $\kappa_1'(c_k)^2$, where κ_1' is a constant which is independent of z and k. Hence, we can take $\kappa_1 = \lambda\kappa_1' \int_{\boldsymbol{R}^2} \eta_M(z)dz$, and obtain (41)

Next, (42) can be proved as follows. By Hölder's inequality we see that

$$(E^\mu[\Big|\, \|\nabla u_{3,k}\|_2^2 - \|\nabla u_3\|_2^2 \Big|^q])^{\frac{1}{q}}$$
$$\leq (E^\mu[\|\nabla u_3\|_2^{2q}])^{\frac{1}{2q}}(E^\mu[\|\nabla u_{3,k} - \nabla u_3\|_2^{2q}])^{\frac{1}{2q}}$$
$$+(E^\mu[\|\nabla u_{3,k}\|_2^{2q}])^{\frac{1}{2q}}(E^\mu[\|\nabla u_{3,k} - \nabla u_3\|_2^{2q}])^{\frac{1}{2q}}. \tag{46}$$

But each term in the above expectation such as $\|\nabla u_3\|_2^2$, $\|\nabla u_{3,k} - \nabla u_3\|_2^2$ and $\|\nabla u_{3,k}\|_2^2$ has an expression by means of multiple stochastic integrals, for example

$$\frac{\|\nabla u_3(\phi_\omega)\|_2^2}{3^2}$$
$$= \int_{\boldsymbol{R}^8}(\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} (J^1(z-z'))^2\eta_M(z)\eta_M(z')J^{\frac{1}{2}}(z-x_1)J^{\frac{1}{2}}(z-x_2)$$
$$\times J^{\frac{1}{2}}(z'-x_1')J^{\frac{1}{2}}(z'-x_2')dzdz')dW_\omega(x_1)dW_\omega(x_2)dW_\omega(x_1')dW_\omega(x_2')$$
$$+4\int_{\boldsymbol{R}^4}(\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} (J^1(z-z'))^3\eta_M(z)\eta_M(z')J^{\frac{1}{2}}(z-x_1)J^{\frac{1}{2}}(z'-x_1')dzdz')$$

$$\times dW_\omega(x_1)dW_\omega(x_1') + 2\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}(J^1(z-z'))^4\eta_M(z)\eta_M(z')dzdz',$$
$$P-a.s.\ \omega\in\Omega. \tag{47}$$

Using the properties of $\hat{\rho}_k(\xi)$ and passing to a standard argument concerning the calculation of the expectation of multiple stochastic integrals (cf. [15] and also Section V.1 of [13]), by (47) and the corresponding expressions through multiple stochastic integrals for the other terms, it is easy to see that there exists C_1 which depends only on M such that

$$(E^\mu[(\|\nabla u_3\|_2^2)^2])^{\frac{1}{2}} \le C_1,$$
$$(E^\mu[(\|\nabla u_{3,k}\|_2^2)^2])^{\frac{1}{2}} \le C_1;$$

also for each $\alpha > 0$ there exists C_2 which depends only on M such that

$$(E^\mu[(\|\nabla u_{3,k} - \nabla u_3\|_2^2)^2])^{\frac{1}{2}} \le C_2 k^{-\alpha}.$$

Since for random variables having multiple stochastic integral representation we can apply Nelson's Hypercontractive bound (cf. Theorem 1.22 of [13]), from the above inequality we can deduce the following:

$$(E^\mu[(\|\nabla u_3\|_2^2)^q])^{\frac{1}{q}} \le (q-1)^2 C_1, \tag{48}$$
$$(E^\mu[(\|\nabla u_{3,k}\|_2^2)^q])^{\frac{1}{q}} \le (q-1)^2 C_1, \tag{49}$$
$$(E^\mu[(\|\nabla u_{3,k} - \nabla u_3\|_2^2)^q])^{\frac{1}{q}} \le (q-1)^2 C_2 k^{-\alpha}, \qquad q = 2,3,\ldots. \tag{50}$$

Then, by (46), (48), (49) and (50) we conclude that there exists some C' that depends only on M such that

$$(E^\mu[|\,\|\nabla u_{3,k}\|_2^2 - \|\nabla u_3\|_2^2|^q])^{\frac{1}{q}} \le (q-1)^2 C' k^{-\alpha}.$$

Moreover using that $\delta u_{3,k}(\phi_\omega)$ and $\delta u_3(\phi_\omega)$ have expressions by means of multiple stochastic integral we easily see that

$$(E^\mu[|\delta u_{3,k} - \delta u_3|^q])^{\frac{1}{q}} \le (q-1)^2 C' k^{-\alpha}.$$

Combining these evaluations we obtain (40).

■

Let $\Lambda_{\lambda u}(\psi)$ be the random variable given in Definition 1.2. Then from Theorem 1.3-1°), for u as in Lemma 2.1 the following holds:

$$\begin{aligned}
&\Lambda_{\lambda u}(\psi)\\
&= \det{}_2\Big(I_{H^{-1}} + 3\lambda\langle \eta_M(\cdot)\,,\, \tau_{(\beta,2)}(\psi)(\cdot)J^{\beta-\frac{1}{2}}(\cdot - x)J^{\beta-\frac{1}{2}}(\cdot - y)\rangle_{S,S'}\Big)\\
&\times \exp\left\{-\lambda\langle \eta_M, \tau_{(\beta,4)}(\psi)\rangle_{S,S'} - \frac{\lambda^2}{2}|J^{\beta-\frac{1}{2}}(\eta_M\tau_{(\beta,3)}(\psi))|_{H^{-1}}^2\right\}.
\end{aligned} \tag{51}$$

Lemma 2.2 *Let $a > \frac{d}{4} - \frac{1}{2}$ and $b > d$. Under the assumptions of Theorem 1.3-2°) the following holds:*

$$\Lambda_{\lambda u} \in \cap_{q<\infty} L^q(\mu), \qquad E^\mu[\Lambda_{\lambda u}] = 1. \tag{52}$$

Let

$$D = \{y \in B_d^{a,b}\,|\, \det{}_2(I_{\mathcal{H}} + \lambda\nabla u(y)) \ne 0\},$$

and let $N(\psi, D)$ denote the cardinality of the set $T^{-1}\{\psi\}\cap D$ for $T(\psi) = \psi + i(\lambda u(\psi))$, then $N(\psi, D)$ is a measurable function and the following holds:

$$\mu(\{\psi\,|\,1 \le N(\psi, D) < \infty\}) = 1. \tag{53}$$

Proof. First of all we recall a crucial result for $H-C^1$ maps on abstract Wiener spaces derived by Kusuoka [7] (cf. also Proposition 3.5.1 of [14]): For a map u that is $H-C^1$ let T be the shift defined by Definition 2.1, then there exists a sequence of measurable sets $G_n \subset B_d^{a,b}$, $n \in \boldsymbol{N}$, such that $\cup_n G_n = D$, and there exists a sequence of shifts T_n, $n \in \boldsymbol{N}$, such that $T_n = T$ *a.s.* on G_n, T_n is bijective and the inverse T_n^{-1} is measurable.

Under the assumptions of Theorem 1.3-2°) since u_3 is an $H-C^1$ map, by this fundamental observation we can consider the properties of such measurable functions $N(\psi, D)$ and $\sum_{y\in T^{-1}(\psi)} \mathrm{sign}(\Lambda_{\lambda u}($ Namely, in theorem 9.3.2 and Remark 9.3.3 of [14] it is shown that if u satisfies (39) then (52) holds. On the other hand, in Theorem 9.2.4 of [14] it is shown that (39) is also a sufficient condition for u under which the following holds:

$$E^{\mu}[\Lambda_{\lambda u}] = \sum_{y\in T^{-1}(\psi)} \mathrm{sign}(\Lambda_{\lambda u}(y)), \qquad \mu-a.s. \quad \psi \in B_d^{a,b}. \tag{54}$$

Since $\Lambda_{\lambda u}(y) = 0$ and $\mathrm{sign}(\Lambda_{\lambda u}(y)) = 0$ for $y \notin D$, by (52) and (54) we see that

$$\sum_{y\in T^{-1}(\psi)\cap \mathcal{D}} \mathrm{sign}(\Lambda_{\lambda u}(y)) = \sum_{y\in T^{-1}(\psi)} \mathrm{sign}(\Lambda_{\lambda u}(y)) = 1 \quad \mu-a.s. \psi \in B_d^{a,b}.$$

By this we have

$$1 \le \sum_{y\in T^{-1}(\psi)\cap D} |\mathrm{sign}(\Lambda_{\lambda u}(y))| = \sum_{y\in T^{-1}(\psi)\cap D} 1 = N(\psi, D)$$

$$\mu-a.s. \quad \psi \in B_d^{a,b}.$$

On the other hand by (52) since $E^{\mu}[|\Lambda_{\lambda u}|] < \infty$, and by Theorem 3.5.2 of [14] since $E^{\mu}[|\Lambda_{\lambda u}|] = E^{\mu}[N(\cdot, D)]$ we have

$$N(\psi, D) < \infty \qquad \mu-a.s. \quad \psi \in B_d^{a,b}.$$

Combining these facts we have (53). ■

Theorem 2. 3 (Solution for the space-cut-off cubic perturbation case) *For given d and $p=3$ take the positive numbers a, a′ and β to satisfy the assumptions of Theorem 1.3-2°). Also take number $\lambda \ge 0$ to satisfy $\lambda < \frac{2}{9L}$, where L is the number defined in Lemma 2.1. For some fixed positive number M let $\eta_M(x) = \eta_1(\frac{x}{M})$ (cf. (6)), and define*

$$T_3(\psi) = \psi + i(\lambda u_3(\psi)), \qquad u_3(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,3)}(\psi)) \tag{55}$$

and

$$d\nu_3 = q\circ T_3 |\Lambda_{\lambda u_3}| d\mu \qquad \textit{for } q \textit{ such that}$$

$$q(\psi) = \begin{cases} \dfrac{1}{N(\psi, D)} & \textit{if } N(\psi, D) \neq 0 \\ 0 & \textit{otherwise,} \end{cases}$$

where $\Lambda_{\lambda u_3}$ is given by (51), and the measurable function $N(\psi, D)$ is defined in Lemma 2.2. Then $\Lambda_{\lambda u_3}\mu$ is a (signed) Girsanov measure and ν_3 is a Girsanov probability measure on $(B_d^{a,b}, \mathcal{B}^{\mu})$ associated with μ and T_3 :

i)

$$E^{\mu}[f\circ T_3 \Lambda_{\lambda u_3}] = E^{\mu}[f], \quad E^{\nu}[f\circ T_3] = E^{\mu}[f] \ \forall f \in C_b(B_d^{a,b}). \tag{56}$$

ii) ν_3 gives a solution of (57) below in the following sense: if ψ is a $B_d^{a,b}$-valued random variable with probability law ν_3, then the following holds for some isonormal Gaussian process W on $\boldsymbol{R}^d$:

$$(-\Delta + m^2)^{1+(\beta-\frac{1}{2})}\psi(x) + \lambda\eta_M(x)\tau_{(\beta,3)}(\psi(x)) = (-\Delta+m^2)^{\beta}\dot{W}(x). \tag{57}$$

Proof of Theorem 2.3. First of all we note that $q(T_3(\psi))|\Lambda_{\lambda u_3}(\psi)|$ can be taken as a $\mathcal{B}^{\mu}$-measurable function: For the $\mathcal{B}^{\mu}$-measurable shift $T_3(\psi)$ with the $H - C^1$ map u_3, since $T_3 *(\mu|D)$ (the image measure of $T_3(\psi)$ restricted to D) is absolutely continuous with respect to μ (cf. Theorem 3.5.2 in [14]), we can define the random variable $q(T_3(\psi))|\Lambda_{\lambda u_3}(\psi)|$ without ambiguity by using a Borel measurable $q(\psi)$ which is defined through any Borel measurable version $\tilde{N}(\psi)$ of $N(\psi)$ such that

$$N(\psi, D) = \tilde{N}(\psi, D) \qquad \mu - a.s. \quad \psi \in B_d^{a,b}$$

(cf. the proof of Lemma 2.2).

Noticing this, by (53) we can apply Corollary 3.5.3 of [14] to our shift T_3, which then yields the results. ■

Remark 2. 2 (Comparison with $(\phi^4)_2$ field) *When $d = 2$ we can take $\beta = \frac{1}{2}$ a case of special interest in Euclidean quantum field theory. In this case the above theorem tells us that the measure ν_3 gives a solution of (35) with space-cut-off:*

$$(-\Delta + m^2)\psi(x) + \lambda\eta_M(x) : \psi^3(x) := (-\Delta + m^2)^{\frac{1}{2}}\dot{W}(x), \; x \in \boldsymbol{R}^2. \tag{58}$$

ν_3 can be written by

$$\begin{aligned}\nu_3(d\psi) = {} & q(T(\psi))\Big|\det{}_2\big(I_{H^{-1}} + 3\lambda\eta_M(x) : \psi^2(x) : \delta_{\{x\}}(y)\big)\Big| \\ & \times \exp\Big\{-\lambda\int_{\boldsymbol{R}^2}\eta_M(x) : \psi^4(x) : dx - \frac{\lambda^2}{2}\int_{\boldsymbol{R}^2}(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2 dx\Big\} \\ & \times\mu(d\psi),\end{aligned}$$

where we have used the fact that $J^{\beta-\frac{1}{2}}(x) = \delta_{\{0\}}(x)$ for $\beta = \frac{1}{2}$ (cf. Theorem 1.3).

On the other hand, the $(\phi^4)_2$ Euclidean field with space-cut-off η_M is a random field on $\boldsymbol{R}^2$ with the probability measure ν_{η_M} such that (cf., e.g., Definition in Section 1 of [13] (pp.141))

$$d\nu_{\eta_M}(\psi) = \frac{1}{Z_M}\exp\{-\lambda\int_{\boldsymbol{R}^2}\eta_M(x) : \psi^4(x) : dx\}d\mu,$$

with the normalization constant $Z_M = E^{\mu}[\exp\{-\lambda\int_{\boldsymbol{R}^2}\eta_M(x) : \psi^4(x) : dx\}]$. Then, there is a similarity between ν_3 and ν_{η_M} in the sense that their Radon-Nikodym densities $\frac{d\nu_3}{d\mu}$ resp. $\frac{d\nu_{\eta_M}}{d\mu}$ have the common term $\exp\{-\lambda\int_{\boldsymbol{R}^2}\eta_M(x) : \psi^4(x) : dx\}$. But, because of the existence of the other non-linear (also non-local) terms in $\frac{d\nu_3}{d\mu}$ such that $q(T(\psi))$, $\Lambda_1 = |\det_2 l(I_{H^{-1}} + 3\lambda\eta_M(x) : \psi^2(x) : \delta_{\{x\}}(y)|$ and $\Lambda_2 = \exp\{\frac{\lambda^2}{2}\int_{\boldsymbol{R}^2}(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2 dx\}$, we have to distinguish ν_3 from ν_{η_M} (as far as $q(T(\psi))$, Λ_1 and Λ_2 do not cancel each other).

We also remark that $(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2$, which is the integrand of Λ_2, is non-local in the sense that $(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2 = (\int_{\boldsymbol{R}^2} J^{\frac{1}{2}}(x-y)\eta_M(y) : \psi^3(y) : dy)^2$ is not measurable with respect to the σ-field generated by the random variable $< \psi(\cdot), \delta^{\epsilon}_{\{x\}} >$ with δ^{ϵ} a $C_0^{\infty}(\boldsymbol{R}^2)$ approximation of the Dirac measure at the point x.

Moreover, since $\int_{\boldsymbol{R}^2}(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2 dx = \int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} J^1(y-y')\eta_M(y)\eta_M(y')(: \psi^3(y) :)(: \psi^3(y') :)dydy'$ and $J^1(y)$ on $\boldsymbol{R}^2$ diverges like "$-\log|y|$" (near 0), it is possible to say that the exponent of Λ_2 contains a term of higher order than $: \psi^4 :$.

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